5.2 (continued)

eigenvalues are complex

__example__ $\vec{x}' = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \vec{x}$

eigenvalues: $\begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0$

$(-1-\lambda)^2 + 4 = 0$

$(-1-\lambda)^2 = -4$ $\qquad -1-\lambda = \pm 2i$

$\boxed{\lambda = -1 \pm 2i}$

complex λ are in complex conjugate pairs

eigenvectors: solve $(A-\lambda I)\vec{v} = \vec{0}$

$\lambda = -1 + 2i$

$$\begin{bmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \end{bmatrix}$$

$$\to \sim \begin{bmatrix} -2 & -2i & 0 \\ -2i & 2 & 0 \end{bmatrix}$$ mult. row 1 by $-i$ add to row 2

$$\to \begin{bmatrix} -2 & -2i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$\lambda = -1 - 2i$

$$\begin{bmatrix} 2i & 2 & 0 \\ -2 & 2i & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

eigenvectors are also conjugate pairs

Solutions are still $e^{\lambda t}\vec{v}$ but usually we want it to not contain imaginary numbers

$$\lambda = -1+2i \quad \vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda = -1-2i \quad \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$e^{\lambda t}\vec{v} = e^{(-1+2i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-t} e^{i(2t)}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Euler's identity:

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-t}\left(\cos(2t) + i\sin(2t)\right)\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \sin(2t) - i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + i e^{-t}\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$$

repeat w/ $\lambda = -1-2i$ $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

conjugate pairs again

$$e^{\lambda t}\vec{v} = \cdots = e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} - i e^{-t}\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$$

real and imaginary parts of either solution are themselves solutions to $\vec{x}' = A\vec{x}$

so, we use them to form the general solution

real part: $e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

imag part: $e^{-t}\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$

general solution:

$$\boxed{\vec{x} = c_1 e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}}$$

$\sin(2t)$ and $\cos(2t)$ are periodic, so x_1 and x_2 are oscillating so phase diagram consists of spirals (into origin if real part of λ is negative

away from origin " " positive)

if real part is 0, ovals

spiral in clockwise direction or counterclockwise

how to figure out ?

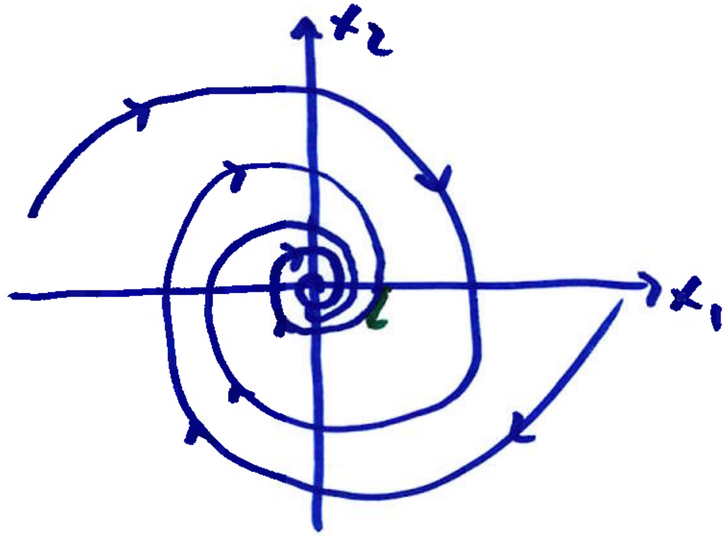
easy way: $\vec{x}\,' = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \vec{x}$

tangent vectors to spirals

pick a convenient $\vec{x}$ then see direction of $\vec{x}\,'$

for example, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \vec{x}\,' = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

left and down



5.5 Repeated Eigenvalues

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} \qquad \lambda = 1, 1$$

algebraic multiplicity is Two

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

linearly independent

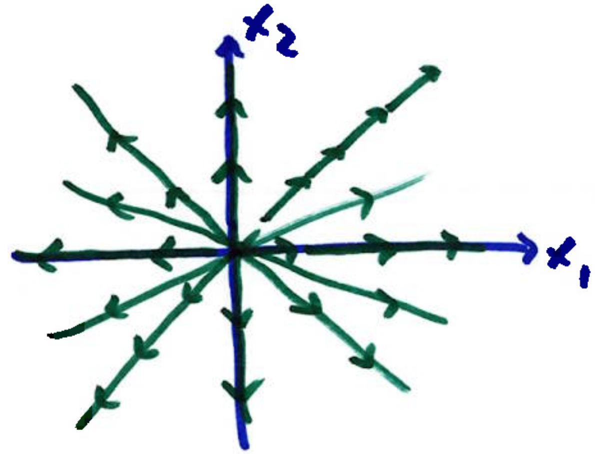
$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

geometric multiplicity is Two

Solutions: $e^{\lambda t}\vec{v}$ $\quad e^{t}\begin{bmatrix}1\\0\end{bmatrix},\ e^{t}\begin{bmatrix}0\\1\end{bmatrix}$

general solution: $\vec{x} = c_1 e^{t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}0\\1\end{bmatrix}$

phase diagram:



now let's look at $\vec{x}' = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}\vec{x}$

$\lambda = 2,\ 2$

Solving $(A-\lambda I)\vec{v} = \vec{0}$

only produces one $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$

missing one eigenvector

$\rightarrow$ matrix A is defective

(here, a defect of one)

one solution: $e^{\lambda t}\vec{v}$

2nd solution: $e^{\lambda t}(t\vec{v}+\vec{u})$ generalized eigenvector

where $(A-\lambda I)\vec{u}=\vec{v}$ $\vec{v}=\begin{bmatrix}1\\0\end{bmatrix}$

here, $\left[\begin{array}{cc|c}0 & 1 & 1\\0 & 0 & 0\end{array}\right]$ $\vec{u}=\begin{bmatrix}a\\1\end{bmatrix}$ choose any a so $\vec{u}$ is linearly indp from $\vec{v}$ and $\vec{u}\neq\vec{0}$

let's use $a=0$, so $\vec{u}=\begin{bmatrix}0\\1\end{bmatrix}$

solutions: $e^{\lambda t}\vec{v}=e^{2t}\begin{bmatrix}1\\0\end{bmatrix}$

$$e^{\lambda t}(t\vec{v}+\vec{u})=e^{2t}\left(t\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right)$$